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Is extensivity a fundamental property of entropy?

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Abstract

The role of linearity in the definition of entropy is examined. While discussions of entropy often treat extensivity as one of its fundamental properties, the extensivity of entropy is not axiomatic in thermodynamics. It is shown that systems in which entropy is an extensive quantity are systems in which a entropy obeys a generalized principle of linear superposition.

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The role of extensivity (this can be thought of as linearity in joining sub-components) in complex compound systems is not axiomatic, as some have supposed [1–3]. This perception has led to controversy as the macrodynamics of physical systems such as black holes has been investigated. An article by Maddox [4] dealing with the question of whether black hole thermodynamics obeys the extensive property has caused us to re-examine the question of whether or not the extensivity of entropy is an intrinsic principle of thermodynamics. As Maddox states '*Everybody who knows about entropy knows that it is an extensive property...*'. In common with the majority of physicists, we were taught that entropy, like mass is extensive. Thus, it is not surprising that the idea that entropy may not always be extensive is resisted by many physicists. (Note, Tsallis [5] has proposed a generalization of entropy so that it is inherently non-extensive. This is not what we are discussing, though there is a role for generalized superposition in a discussion of Tsallis entropy.) It is this proposition and its wider consequences that we discuss herein.

Entropy was defined phenomenologically by Clausius as the integral of $\frac{\mathrm{d}Q}{T}$ over a reversible path. Reif [3] argues that since the heat absorbed $\mathrm{d}Q$ is extensive, then entropy is extensive as a consequence. Jaynes [6] has pointed out that the Clausius definition says nothing about extensivity since the size of the system does not vary over the integration path. The extensivity of entropy is an additional condition, imposed separately.

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If extensivity is not contained in the definition of entropy, are there situations where entropy is not an extensive quantity? Non-extensive entropies can be found in a variety of other situations. Systems subjected to long-range forces are found to have non-extensive entropies if the potential energies associated with these forces are not negligible [7]. The Coulomb force gives rise to non-extensive entropies for aggregates of charged particles carrying a net charge [8]. Gravity causes non-extensive entropies in astrophysical situations. A common example of a system with a non-extensive entropy is a column of liquid in which gravity has induced a density gradient. Entropy is also non-extensive in small thermodynamic systems where edge effects are important [9]. This catalog of systems with non-extensive entropies is representative rather than exhaustive. We conclude that while entropy is extensive in many situations, extensivity is not a fundamental property of entropy. Now, having discussed a variety of sources that have demonstrated that entropic extensivity cannot be one of the fundamental axioms of thermodynamics, we explore the meaning of extensivity and gain insight into when it is useful.

Insight into the usefulness of extensivity can be developed by relating it to the basic results of the theory of generalized linear systems. Consider a system with inputs $x_i(n)$, scalars c_i and a system transform $T()$. This system will be linear provided the system obeys the principle of superposition (+ means ordinary addition)

$$T\,[x_1(n) + x_2(n)] = T\,[x_1(n)] + T\,[x_2(n)] \tag{1}$$

and scalar multiplication $\cdot$

$$T\,[c \cdot x(n)] = c\,T\,[x(n)]\,. \tag{2}$$

In order to generalize what is meant by linear systems to other non-linear systems with a transform $H()$, Oppenheim [10] proposed that the summation sign must be replaced with two different symbols: a rule for combining inputs by $\circledast$, and a different rule for combining outputs by using the symbol $\oplus$. We then write the generalized superposition principle for such a system as

$$H\,[x_1(n) \circledast x_2(n)] = H\,[x_1(n)] \oplus H\,[x_2(n)] \tag{3}$$

which is the generalization of (1). A similar condition can be defined for scalar multiplication by a constant c,

$$H\,[c \otimes x(n)] = c \circ H\,[x(n)] \tag{4}$$

where $\otimes$ replaces the input scalar multiplication and $\circ$ replaces the output scalar multiplication. (Note that this notation is slightly in variance with the literature [11].) Systems that have inputs and outputs that satisfy both (3) and (4) are referred to as *homomorphic systems* since they can be represented as algebraically linear (homomorphic) mappings between the input and output signal spaces. (Note, this formalized view of homomorphic transformations can be relevant in a number of other physical theories including as quantum mechanics.)

One would like to know if a given nonlinear combination of functions can be interpreted as obeying a generalized superposition principle. To do this, one must be able to determine if the nonlinear combination of functions shown in figure 1 can be separated into a linear combination by application of a separator function S. For a system defined as

$$x(n) = [x_1(n)] \cdot [x_2(n)] \tag{5}$$

the separator function for multiplication is the logarithm, so that

$$\log\,[x_1(n)] \cdot [x_2(n)] = \log\,[x_1(n)] + \log\,[x_2(n)] \tag{6}$$

and we have the appearance of superposition. Most nonlinear functions $N(x,\,y)$ do not separate in a manner that a superposition principle holds; e.g. a separator function S does not exist such

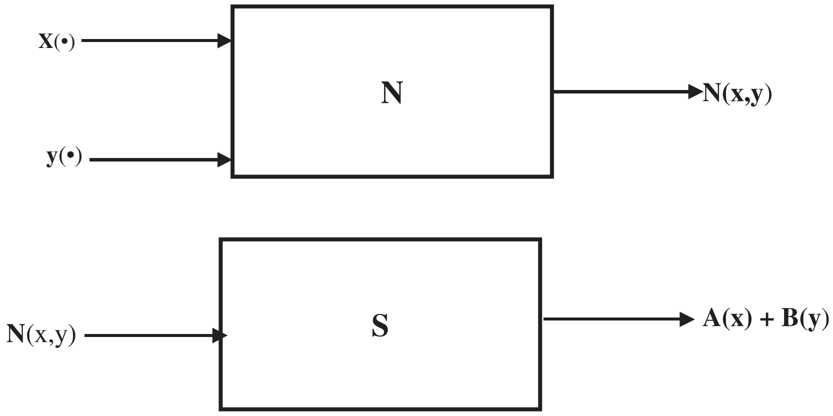

Figure 1. Nonlinear combination of two signals (top) and separator function and separands (bottom).

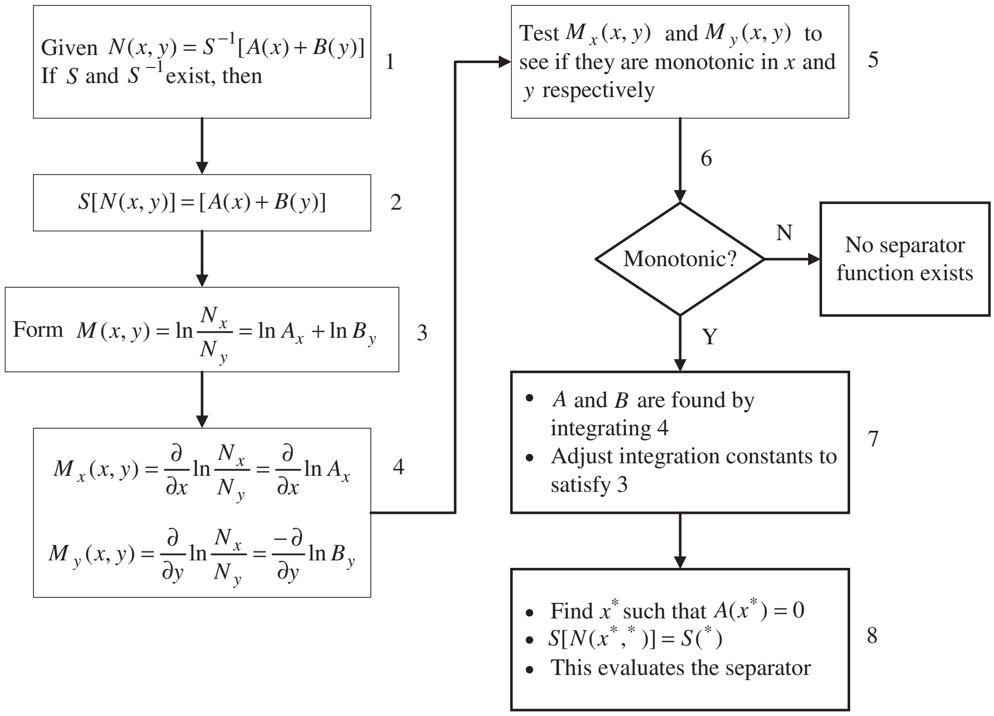

Figure 2. Flow chart of the mathematical details for determining if the rule for combining systems can be deconstructed so that a valid separator function S can be found (based on [12]).

that $S[N(x, y)] = A(x) + B(y)$. It is a non-trivial exercise to find whether other rules for combining of systems have separator functions. The mathematical details for determining if the rule for combining systems can be deconstruct so that a valid separator function S can be found are shown in the flowchart in figure 2 which is based on [12].

When we say that entropy or any other physical function is extensive, we are saying that it obeys a generalized superposition principle. Thus, generalized superposition provides an interpretation of the extensivity property of entropy when it exists. For example, subcomponents of a system are combined so that the statistical weight of the combined system is the product of the statistical weights of the individual components, we can cast this into the formalism of generalized superposition. By using the generalized superposition expression, we see that if the system transform is the natural logarithm ln, $\circledast$, and $\oplus$ represent addition and multiplication respectively, then,

$$\ln \Omega_1 \Omega_2 = \ln \Omega_1 + \ln \Omega_2 \tag{7}$$

and we see that the Boltzmann definition of entropy enables the introduction of a quantity that is additive over sub-systems. Thus, we now see the role played by extensivity more clearly. In fact, generalized superposition can be used as a guiding principle to look for extensive variables in a generalized setting [13, 14] of trying to determine the thermodynamics of complex systems. More general types of entropy-like variables can be defined using a form of generalized superposition[4].

The principle of linear superposition plays a major role in the analysis of wave fields, we are accustomed to its failure when we analyze finite amplitude waves and in the high field case. We contend that the extensivity of entropy should be viewed similarly. Entropy is approximately extensive in many situations, and we should continue to exploit this property when it is convenient. Given the rule for combining subsystems into the whole is nonlinear, the principle of generalized superposition allows us to look for a rule that gives the appearance of linearity and hence extensivity. Linearity on a macroscopic scale gives us a generalized thermodynamics of complex systems. However, this need not be the case for entropy since it is known that there are a number of situations where entropy is not extensive. We should cease to consider extensivity as one of the axioms of thermodynamics, and should instead consider it as derivative of situations in which a principle of generalized superposition can be found.

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[4] It is our hypothesis that part of the phenomena of emergence is due to counting functions that enumerate the states that obey this generalized form of superposition or a form of linearization This need not be strictly linear either, (so we call it Stirling linear) as is illustrated by attempting to linearize a counting function such as the gamma function ($n!$). While $\ln(n!)$ is not linear, it is approximately so for a physical system with a number of components. For all practical purposes $\ln((n + m)!) \approx n \ln(n) + m \ln(m) \approx \ln(n!) + \ln(m!)$, so $n!$ has been linearized with respect to the logarithm. The difference between approximately linear and linear is a point that is not emphasized as the gamma function illustrates. Delineated linear in the sense of Stirling would be a step in clarifying this issues of what entropy and complexity really are in a broader macrodynamics. One could then argue a non-physical form of entropy exists only a linearization (Stirling) principle exists and use this to formulate an axiomatic basis for an underlying 'thermodynamics'.

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